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### Abstract

Optimal design of multilaminar fibre-reinforced structures presents interesting problems. In this paper, the problem is formulated as that of first seeking the deflection pattern associated with the optimal structure, and then inferring the optimal design from this. This results in a decomposition of the problem, and allows algorithms to be designed which simultaneously determine the optimal number of layers in each finite element, their thicknesses and their fibre directions. In addition, formulation in these terms allows some useful insights to be gained without computation. The paper includes a description of algorithms and numerical results.

### 1. Introduction

The introduction of highly directional materials such as boron and carbon fibre reinforced plastics poses interesting and difficult problems for structural designers. The great difference in strength and stiffness along and transverse to the fibre axis means that the full benefits of these materials can only be realised by near-optimum design; the penalties for poor design can be correspondingly high.

These characteristics mean that the design of such structures is a natural area for the application of numerical optimisation techniques. However, the problem posed is certainly more difficult than its counterpart in the design of isotropic structures; in addition to the distribution of material, it is now necessary to determine the best layup of the fibres. The latter requirement introduces a highly nonlinear element into the problem.

This paper is concerned with the design of a particular class of structures, namely those constructed from two dimensional plane stress elements, each composed of a number of layers. In each layer the fibres are unidirectional, and the layers are distinguished from one another in having different directions for their fibre axes. Three-dimensional structures composed of such two-dimensional elements are of course common in aircraft design practice. The design problem is to determine, for each point in the structure, the optimal values of the number of layers, their thicknesses, and their fibre directions so as to minimise the overall volume subject to stress and displacement limits under a number of alternative applied load sets.

The simpler isotropic version of this problem involves only material thicknesses, and has been studied seriously for many years. Even this however, is far from trivial<sup>(1,2)</sup>, and even at this date it is probably fair to say that no completely satisfactory method for its solution is in general use. The essential difficulty lies in the nonlinear relationship between the element thicknesses and the strains in the structure. This means that the structural optimisation problem has nonlinear and in fact nonconvex constraints which, combined

with the large number of variables in the problem means that most of the commonly used methods of nonlinear programming require a considerable consumption of computing resources to solve practical problems. Although new methods of nonlinear programming such as Recursive Quadratic Programming and Augmented Lagrangian techniques will eventually have an impact, the current situation seems to be that fairly ad-hoc methods have had to be employed in everyday use<sup>(3)</sup>.

Given this background, the difficulty involved in introducing additional variables into the problem (some nonlinear, some integer) is plain to see. One approach<sup>(4)</sup> has been to restrict the allowable fibre angles to a small discrete set. This reduces the problem to a quasi-isotropic one, with more thickness variables. Such an approach has much to recommend it from an immediately practical point of view; restrictions must be imposed in any case upon the fibre angles to facilitate manufacture, and the simplification allows the problem to be solved by available techniques. However, there is clearly a need for an approach which is capable of solving the problem as it stands. Such a method would allow restrictions on fibre angles to be imposed as required, rather than having them imposed willy-nilly; and in addition, from the point of view of long-term progress, it is desirable to have methods which exploit the particular properties of these structures rather than forcing them arbitrarily into a conventional framework.

The present paper describes a new formulation of the structural optimisation problem which seems to be peculiarly suited to multilaminar composites. Its basis is a change of variables; the idea is to use the nodal displacement of a finite element model as the optimisation variables, rather than the more usual design variables. The problem becomes in effect one of searching for optimal deflection patterns under the applied loads; the subsidiary problem of finding the structures associated with such deflection patterns is imbedded in the overall algorithm. The benefits flowing from this reformulation are the following: (1) an immediate insight into the nature of multilayer composites, particularly as concerns the number of layers necessary; (2) a decomposition of the problem into two, one having a structure sufficiently similar to that of Linear Programming to allow an algorithm to be designed which solves the integer programming aspect without restriction; (3) the capacity to determine simultaneously the optimal number of layers, their thicknesses and their fibre angles.

### 2. A deflection - space formulation

In this section the general problem described in the introduction will be restricted to exclude both direct constraints on stress, and multiple load cases; we shall, however, retain constraints on the deflections. These restrictions will be relaxed in later sections.

Let the following vectors of design variables be

introduced:

$$\underline{t} \equiv \{t_1^1, t_2^1, \dots, t_{L_1}^1, t_1^2, \dots, t_{L_N}^N\}$$

$$\underline{\theta} \equiv \{\theta_1^1, \theta_2^1, \dots, \theta_{L_1}^1, \theta_1^2, \dots, \theta_{L_N}^N\}$$

$$\underline{L} \equiv \{L_1, \dots, L_N\}$$

where  $t_j^i$  is the thickness of the  $j$ 'th layer in the  $i$ 'th finite element and  $\theta_j^i$  the corresponding fibre angle;  $L_i$  is the number of layers in the  $i$ 'th finite element.  $N$  is of course, the total number of such elements. Any design  $D$  is completely defined once values have been assigned to  $\underline{t}$ ,  $\underline{\theta}$  and  $\underline{L}$ .

The overall problem to be solved can now be expressed as follows:

$$\begin{aligned} \text{Min } W &= \sum_{i=1}^N A_i \sum_{j=1}^{L_i} t_j^i \\ \underline{t}, \underline{\theta}, \underline{L} \quad \text{s.t. } f^k(\underline{\delta}) &\leq 0, \quad k=1, 2, \dots, q \\ \underline{\delta} &= \underline{K}^{-1} \underline{P} \\ t_j^i &\geq 0 \\ \theta_j^i &\in H^i \\ L_i &\text{ integer} \end{aligned} \quad (2.1)$$

$H^i$  is the set of allowable angles for the layers in element  $i$  - usually the half - open interval  $[0, \pi)$ . The quantity  $\underline{\delta}$  is the deflection vector, defined as follows:

$$\underline{\delta} \equiv \{\delta_1, \delta_2, \dots, \delta_M\}$$

where  $M$  is the (fixed) number of nodal degrees of freedom of the structure. The deflections are due to a fixed load vector  $\underline{P}$ , also of dimension  $M$ .  $A_i$  is the (fixed) plane area of the  $i$ 'th finite element. The scalar functions  $f^k(\underline{\delta})$  may be of any form.

The deflections are related to the loads through a stiffness matrix,  $\underline{K}$ , which in turn depends upon the design variables:

$$\underline{\delta} = \underline{K}^{-1}(\underline{t}, \underline{\theta}, \underline{L}) \underline{P} \quad (2.2)$$

$\underline{K}$  is linear in  $\underline{t}$  but highly nonlinear in  $\underline{\theta}$ , so that its inverse is nonlinear in both of these sets of variables.

Before considering the formulation of problem (2.1) in terms of deflections, it is convenient to state the following optimality condition:

Let  $D^*$  be a design which solves problem (2.1) and let  $\underline{\delta}^*$  be its vector of deflections under load  $\underline{P}$ . Then, of all allowable structures having this set of deflections under the load  $\underline{P}$ ,  $D^*$  is that which has minimum volume. An allowable design is one which satisfies the direct constraints on the design variables in problem (2.1).

The proof of this is straightforward, and will not be repeated here. The property can be regarded as a necessary condition on  $D^*$ , given  $\underline{\delta}^*$ .

Consider now the way in which problem (2.1) must be manipulated in order to express it in terms of  $\underline{\delta}$  only. The deflection constraints are of course already in a suitable form, but the objective function poses a difficulty. In the form given, this function may be regarded as a means of associating uniquely a cost  $W$  with a design  $D(\underline{t}, \underline{\theta}, \underline{L})$ . Ideally, therefore, one would wish to associate a unique design  $D$  with every deflection  $\underline{\delta}$  under the load  $\underline{P}$ . Although (in the absence of instability) each  $D$  has a uniquely defined  $\underline{\delta}$ , the converse is hardly ever true. However, the property stated above provides a means of associating a unique volume with every deflection. It is only necessary to use the fact that the optimum design  $D^*$  is uniquely related to its deflection  $\underline{\delta}^*$  in that, of all allowable designs having that deflection under  $\underline{P}$ , the design  $D^*$  is of minimum volume. If this necessary condition is enforced at every value of  $\underline{\delta}$  under consideration, an objective function  $W(\underline{\delta})$  can be defined as follows:

Let  $\underline{\delta}$  be any  $M$ -vector; then  $W(\underline{\delta})$  is defined as the volume of that structure which has a deflection equal to  $\underline{\delta}$  under the load  $\underline{P}$ , which is allowable, and which is of lower volume than any other structure satisfying these conditions. If no such structure exists, then  $W(\underline{\delta})$  is of infinite value. Problem (2.1) becomes:

$$\begin{aligned} \text{Min } W(\underline{\delta}) \\ \text{s.t. } f^k(\underline{\delta}) &\leq 0 \quad (P1) \\ k &= 1, 2, \dots, q \end{aligned}$$

It is clear that, in order to evaluate  $W(\underline{\delta})$ , it is necessary to solve a subsidiary minimisation problem, and thereby to generate a design (or designs)  $D(\underline{\delta})$  which satisfies the necessary condition. Evidently this problem (the inner subproblem) can be expressed as follows:

$$\begin{aligned} \text{Min } W(\underline{\delta}) &= \sum_{i=1}^N A_i \sum_{j=1}^{L_i} t_j^i \\ \underline{t}, \underline{\theta}, \underline{L} \quad \text{s.t. } \underline{K}(\underline{t}, \underline{\theta}, \underline{L}) \underline{\delta} &= \underline{P} \quad (P2) \\ t_j^i &\geq 0, \theta_j^i \in H^i, L_i \text{ integer} \\ &\text{(where } \underline{\delta} \text{ is of course given).} \end{aligned}$$

The equality constraints in P2 are of course the equilibrium-compatibility conditions for the structure.

Problem (2.1) has now been decomposed into two subproblems. The outer problem is P1, and to solve it a search is required in the  $M$ -dimensional space of the deflections. During this search, the subproblem P2 must be solved each time the objective function  $W(\underline{\delta})$  is evaluated. P2 is in terms of the original design variables, and is a mixed integer problem. At this stage, therefore, it may not be obvious that the problem has been made more tractable by the reformulation. Clearly, P1 does not directly involve integer variables, but the constraints have been simplified at the expense of the objective function. In fact, in many practical problems the function  $f^k(\underline{\delta})$  will be very simple indeed - often only upper and lower bounds on  $\underline{\delta}$ . However,  $W(\underline{\delta})$  is now an unknown quantity as far as its functional behaviour is concerned. In addition, the subproblem P2, which will have to be solved

repeatedly in the course of solving P1, is on the face of it a mixed integer problem not very much easier to solve than the original problem itself. Note that the equality constraints are nonlinear in  $\underline{\delta}$ .

It will be the aim of the next section of the paper to show that in fact problem P2 can be solved without difficulty by an algorithm which is a special generalisation of the Simplex algorithm of Linear Programming. Furthermore, the function  $W(\underline{\delta})$  will be shown to possess useful properties from the point of view of solving P1, particularly with regard to the very important class of maximum stiffness structures.

### 3. The inner subproblem P2

The inner subproblem will be considered first because upon the ease of its solution depends the feasibility of the whole approach. We begin by writing the constraints explicitly in terms of the design variables, bearing in mind that the displacement vector  $\underline{\delta}$  is given. Let  $\underline{k}^{i,j}$  be the stiffness matrix of the  $j$ 'th layer in the  $i$ 'th finite element, referred to some global cartesian axes and scaled to unit thickness. Then it is shown in ref. (5) that this matrix can be expressed as follows in the case of fibre-reinforced composites:

$$\underline{k}^{i,j} = \underline{k}_0^i + \underline{k}_1^i \cos 4\theta_j^i + \underline{k}_2^i \sin 4\theta_j^i + \underline{k}_3^i \cos 2\theta_j^i + \underline{k}_4^i \sin 2\theta_j^i \quad (3.1)$$

where  $\underline{k}_r^i$ ,  $r = 0, 1, \dots, 4$  are matrices which depend only on the fixed geometry of the  $i$ 'th element and  $\theta_j^i$  is the angle between the fibre axis in the  $i, j$ 'th layer and the positive  $x$ -axis.

Using (3.1) we can write the total stiffness matrix of the structure as:

$$\underline{K} = \sum_{i=1}^N \sum_{j=1}^{L_i} \left\{ \underline{k}_0^i + \underline{k}_1^i \cos 4\theta_j^i + \underline{k}_2^i \sin 4\theta_j^i + \underline{k}_3^i \cos 2\theta_j^i + \underline{k}_4^i \sin 2\theta_j^i \right\} t_j^i \quad (3.2)$$

The equality constraints in P2 can be written:

$$\begin{aligned} \underline{K} \underline{\delta} &= \sum_{i=1}^N \sum_{j=1}^{L_i} \left\{ \underline{k}^{i,j} t_j^i \right\} \underline{\delta} \\ &= \sum_{i=1}^N \sum_{j=1}^{L_i} \left\{ \underline{k}^{i,j} \underline{\delta} \right\} t_j^i \\ &= \sum_i \sum_j t_j^i \underline{p}_j^i (\theta_j^i) = \underline{P} \end{aligned} \quad (3.3)$$

where  $\underline{p}_j^i (\theta_j^i)$  is the load vector (per unit of thickness) necessary to cause the  $i, j$ 'th layer to deflect by an amount  $\underline{\delta}$ , if it were isolated from the remainder of the structure. Using (3.1) it can be written explicitly as:

$$\underline{p}_j^i = \underline{p}^{i,j} + \underline{p}_1^{i,j} \cos 4\theta_j^i + \underline{p}_2^{i,j} \sin 4\theta_j^i + \underline{p}_3^{i,j} \sin 2\theta_j^i + \underline{p}_4^{i,j} \cos 2\theta_j^i \quad (3.4)$$

where  $\underline{p}_r^{i,j} = \underline{k}_r^{i,j} \underline{\delta}$  ( $r=0, \dots, 4$ ), can be evaluated immediately from the known  $\underline{k}_r^{i,j}$  and  $\underline{\delta}$ . The inner subproblem can now be written in full as follows:

$$\begin{aligned} \text{Min } & \sum_{i=1}^N A_i \sum_{j=1}^{L_i} t_j^i \\ \text{s.t. } & \sum_{i=1}^N \sum_{j=1}^{L_i} t_j^i \underline{p}_j^i (\theta_j^i) = \underline{P} \end{aligned} \quad (3.5)$$

$$t_j^i \geq 0, \theta_j^i \in H^i, L_i \text{ integer.}$$

Problem (3.5) has certain features in common with the classical linear programming problem; indeed, it can be viewed as such but with constraint coefficients which are nonconvex separable functions of variables  $\theta_j^i$  whose optimal values  $\theta_j^i$  are required to be found, in addition to those of the linear variables  $t_j^i$ .

In ref. (6) the author has described a special algorithm for solving this problem, a detailed description of which is not possible within the limits of the present paper. Briefly, it proceeds as follows. It can easily be shown that a design which solves problem 3.5 cannot have more than  $M$  layers in total throughout the structure; the problem therefore is to determine how these layers are distributed and what their thicknesses and fibre angles should be. The algorithm, which will be referred to as the Functional Linear Programming (FLP) algorithm, automatically generates an initial design having  $M$  layers and satisfying the constraints in problem 3.5. Such a design is referred to as 'basic feasible', a term borrowed from classical L.P. practice. The procedure then continues iteratively, computing on each iteration the parameters of a new layer to be introduced in some element so as to decrease the total cost, and determining which layer already present should be discarded so as to maintain basic feasibility. It will be recognised that this logic resembles that of classical L.P.; indeed, the main difference between the two algorithms is that in the case of F.L.P. the reduced gradients, whose values determine the choice of new basic variables, are not scalars but functions of the  $\theta_j^i$ . Therefore on every iteration a number of one-dimensional minimisations are required. This poses no difficulty because the functions to be minimised are simple enough not to require an iterative procedure for their solution.

It will be clear from this synopsis of the algorithm that the optimal numbers of layers in each element (the  $L_i$ ) are generated automatically during the process of adding and discarding layers. Note that on any iteration every element is considered as a candidate to receive a new layer or to lose one which it already has; thus a layer may be added to element  $i$  and another dropped from element  $j$ , where  $i$  may or may not equal  $j$ . It follows that no integer variables as such appear in the problem. Again, because the fibre angles appear as arguments of the reduced gradient functions their optimal values can be found exactly subject only to the fact that such values must be in the sets  $H^i$ , which can be either continuous or discrete. If any of the feasible sets are continuous, the algorithm generates an infinite sequence of solutions which must be terminated when some

convergence criterion has been satisfied. In this it contrasts with classical Linear Programming, which is a finite process.

Having described the general approach to solving problem P2, it is now possible to relax a restriction imposed during the previous section, namely the exclusion of stress constraints. Bearing in mind that we are solving a problem for fixed deflection, it follows that all feasible solutions to P2 have identical strains. The stresses, of course, depend in addition upon the fibre angles  $\theta_i^j$ . Now the way in which these angles are chosen by the algorithm has already been described; they are the values which give minimal reduced gradients at each iteration. Any restriction on stress, therefore, simply means that the one-dimensional minimisations become constrained. There is no direct effect on the thickness variables and, in particular, the total numbers of layers in the structure remain unchanged (see section 5 below).

References 5 and 7 contain a description of the numerical experience which has been gained with the FIP algorithm, implemented as a FORTRAN program. Problems solved have been fairly small because of limitations imposed by computer size; typically M and N would be between 30 and 45. No difficulties have been encountered in any of the cases. Perhaps the most striking result has been the enormous range of designs which can be found, all having identical deflections under a given load. The point is clearly demonstrated by the second of the examples given at the end of the paper. Here, the range was wide enough to include structures whose costs (in this case, volumes) differed from one another in the ratio 10:1.

It was stated at the beginning of this section that the feasibility of the whole deflection - variable approach depends mainly upon the ease with which P2 can be solved. The functional linear programming algorithm provides a means of overcoming the difficulties inherent in the problem, particularly its integer programming aspect, and is both reliable and reasonably efficient. Because this algorithm is in itself a considerable departure from other methods of constrained minimisation, it seems to offer considerable scope for further development both theoretically and numerically. For the moment, however, it can be claimed that the numerical experience so far gained, and described in part in references 5, 6 and 7 and in section 6 of this paper provide a sufficiently firm base to justify proceeding to solve the outer problem P1.

#### 4. The P1 subproblem

The solvability of the outer problem P1 depends both on the properties of the objective function  $W(\underline{\delta})$  and on the particular form of the constraints  $f^k(\underline{\delta})$ . In practice deflection constraints are seldom complicated, and in fact a very important class of structures will be shown to be characterised by only one linear equality constraint. The key question therefore involves the properties of the function  $W(\underline{\delta})$ .

It has been stated in section 3 that  $W(\underline{\delta})$  can be fairly easily evaluated using the Functional Linear Programming algorithm. Clearly, P2 will not have a solution for every vector in M-space;

for example the whole half-space defined by the inequality  $P^t \underline{\delta} \leq 0$  consists of a set of deflections for which any structures would have non-positive strain energy; such a structure would violate the principle of conservation of energy. By definition the value of  $W(\underline{\delta})$  in that half-space is infinite. However, even in the positive-energy half-space there may exist regions in which no structure is defined; all such regions will be described as "physically infeasible" to distinguish them from the merely infeasible regions for which the arbitrary constraints  $f^k(\underline{\delta}) \leq 0$  are not satisfied. Clearly, if P2 can be solved for some deflection  $\underline{\delta}'$ , then a solution also exists for any deflection  $\alpha \underline{\delta}'$ ,  $\alpha > 0$ ; in fact the designs are the same for all positive values of  $\alpha$ , except that the thickness vector  $t$  and therefore the value of  $W$  is scaled by  $1/\alpha$ . It follows that the boundaries of the physically infeasible regions are straight line generators passing through the origin in  $\underline{\delta}$ -space. The existence of such boundaries is a potential source of difficulty with the method, and the behaviour of the function near them requires more investigation. To date, however, they have caused no computational difficulties. It is shown in ref. 5 that  $W(\underline{\delta})$  is continuous and differentiable everywhere except in and on the boundaries of this physically infeasible region.

The differentiability of  $W(\underline{\delta})$  is a very useful feature from the point of view of solving P1. In fact the derivatives are easily computed in spite of the fact that  $W(\underline{\delta})$  is not of explicitly known analytic form. To see how this is done, consider the dual problem associated with P2. This can be written (5,8):

$$\begin{aligned} & \text{Max } P^t \lambda \\ & \underline{\theta}, \lambda \\ & \text{s.t. } P_i^t (\theta^i) \lambda \leq A_i \quad i=1, 2, \dots, N \quad (4.1) \\ & \theta^i \in H^i \end{aligned}$$

where  $\lambda$  is the M-vector of dual variables (Lagrange Multipliers) associated with the equality constraints in (3.5). Either by direct differentiation of (3.5), or by noting that  $\lambda$  can be interpreted as the derivative of the objective function in (3.5) with respect to changes in the right-hand sides of the constraint equations, we obtain the following expression for the derivatives of  $W(\underline{\delta})$ :

$$\frac{\partial W(\underline{\delta})}{\partial \underline{\delta}} = -K \lambda \quad (4.2)$$

where  $K$  is the stiffness matrix of the design which is the optimal solution to (4.1) for the given vector  $\underline{\delta}$ . Since the evaluation of  $W(\underline{\delta})$  involves the solution of P2 and hence the determination of  $\lambda$ , it follows that equation 3.2 allows us to compute the derivatives of  $W(\underline{\delta})$  with almost no additional effort. The usefulness of this property can be judged from the fact that to evaluate the derivatives approximately using central differences would require  $2M+1$  solutions of P2 where  $M$ , the number of nodal degrees of freedom, might well be large.

The investigation of the properties of the function  $W(\underline{\delta})$  requires much more space than it can be

given in a condensed account such as this; ref. 5 contains a more detailed description. However, some observations will be made.

If  $\lambda$  is interpreted as a virtual deflection then the dual P2 (4.1) can be seen as the problem of maximising the total virtual energy under the requirement that the individual strain-energy densities  $\underline{p}_i^t(\theta^i)\lambda/A_i$  be less than unity. (Note that  $\underline{p}_i(\theta^i)$  is a unit thickness force vector, so that the strain-energy density interpretation is dimensionally correct). Taking this a little further, and recalling the definition of  $\underline{p}_i(\theta^i)$  embodied in equations 3.3, it can be seen that the typical constraint of dual P2 takes on the following interesting bilinear form in terms of  $\underline{\delta}$  and  $\lambda$ :

$$\underline{\delta}^t \underline{k}_i(\theta^i)\lambda/A_i \leq 1.$$

This form suggests the question: what is the significance of the special case where  $\underline{\delta}$  is proportional to  $\lambda$ ? In fact, this relationship is a property of maximum stiffness structures, which will be discussed in section 6.

### 5. Upper limits on the number of layers in the optimal structures

Before proceeding to discuss the design of an algorithm for solving the overall problem, it is worthwhile considering the information which can be gleaned from the form of problems P1/P2. Note first that the overall solution to problem (1) must also be a solution to P2. But P2 is essentially a Linear Program in an infinite number of variables; the number of constraints is a finite number  $M'$ . In the case so far considered,  $M'$  is simply the number of degrees of freedom of the structure. Each additional alternative load set increases the number of constraints in P2, although not by  $M$ . If  $(\underline{p}_i, \underline{\delta}_i)$  are the  $i$ 'th alternative load set and its corresponding deflection (for any design) then, by the Virtual Work Theorem:  $\underline{p}_i^t \underline{\delta}_i = \underline{P}^t \underline{\delta}_i$ ,  $i=1,2,\dots,Q$  where  $Q$  is the number of load cases. It follows from this that if the load sets are arbitrarily ordered 1 to  $Q$ , then the first introduces  $M$  constraints into P2, the second ( $M-1$ ), the third ( $M-2$ ) and so on. The maximum number of load cases which can independently influence the design is thus  $M(M+1)/2$ , which is equal to the number of independent elements in the stiffness matrix. If  $Q$  is less than this value, then the number of constraints on P2 is equal to  $MQ - Q(Q-1)/2$ . The significance of this value is, that for any Linear Program, the solution has at most  $M'$  nonzero values of its variables, where  $M'$  is the number of constraints. This property holds also for problems of the form P2, where the corresponding variables are the  $t_j$ . It follows that there will not be more than  $\min(MQ - Q(Q-1)/2, M(M+1)/2)$  layers in total throughout the structure<sup>9</sup>.

This argument has been carried further in 9, where it is shown that from this point of view the finite elements can be considered in isolation. It follows that, for example, a six node element having nine degrees of freedom cannot have more than nine layers if it is to form part of a structure which is optimal under a single load set. Similarly, a three-node element will not have more than three layers. The implications of this for the best choice of idealisation seem interesting.

It can be seen that the formulation of problem

(1) in terms of P1 and P2 immediately yields information which is not so easily available otherwise. It is perhaps worth stressing that the limits obtained above are not imposed as a result of the re-formulation; problems P1 - P2 are together exactly equivalent to (1), and any characteristics of a design which solves them are also those of solutions to 2.1. In fact, the results obtained above are particular cases of a more general theorem applying to any linear structure under any combination of stress and displacement constraints, but excluding buckling constraints. It can be used, for example, to show that the optimal truss under one load set is statically determinate and also to derive the corresponding conditions for multiple alternative loads.

There is another result which can be obtained from similar considerations, but this time it takes account of the particular properties of laminates. It is described in Ref. 9 and summarised here. If we consider any finite element, we can write down a problem of the same form as P2 but involving only that element. The problem cannot of course be solved numerically, because it involves a load vector which depends on the overall applied load and the overall design and is therefore not in general a known quantity. However, by considering the form of this problem it becomes clear that the maximum number of independent constraints on it is in fact five; this number follows from the fact that  $\underline{P}_i$  can be represented (3.4) by a linear combination of five independent vectors and not more. Thus, from the argument used in the first part of this section the number of layers in any element cannot exceed five in an optimal structure. This result, unlike the earlier one which it supplements, is independent of the number of degrees of freedom of the element, and is therefore independent of the idealization.

### 6. Maximum stiffness structures

In this section an important class of problems defined by a particular deflection constraint will be considered. A structure of maximum stiffness is defined to be one whose strain energy is a minimum for a given volume, that is, one whose strain energy density is a minimum. It can be shown<sup>(7)</sup> that such a design is the same as that whose volume is minimal for a given strain energy.

It follows that the maximum-stiffness problem can be expressed as follows:

$$\begin{aligned} \text{Min } & W(\underline{\delta}) \\ \text{s.t. } & \underline{P}^t \underline{\delta} = E \end{aligned} \quad (6.1)$$

Where  $W(\underline{\delta})$  is of course the function defined in section 2 and  $E$  is a fixed energy constraint. Thus the problem has only one constraint, and that linear.

The Lagrangian function associated with problem (6.1) is clearly:

$$L(\underline{\delta}, \mu) = W(\underline{\delta}) - \mu(\underline{P}^t \underline{\delta} - E) \quad (6.2)$$

where  $\mu$  is a Lagrangian multiplier.

For a solution to (6.1) we require that the following condition be satisfied:

$$\nabla L(\underline{\delta}, \mu) = \nabla W(\underline{\delta}) + \mu \underline{P} = 0 \quad (6.3)$$

Using (4.2), together with the equations relating  $\underline{P}$  and  $\underline{\delta}$  we obtain:

$$\underline{K} (\mu \underline{\delta} - \underline{\lambda}) = 0 \quad (6.4)$$

Here, the matrix  $\underline{K}$  as well as the vector  $\underline{\delta}$  and  $\underline{\lambda}$  are those associated with the structure/s satisfying (6.1). Unless  $\underline{K}$  is singular equation (6.4) implies:

$$\mu \underline{\delta} = \underline{\lambda} \quad (6.5)$$

Premultiplying (6.5) by  $\underline{P}^t$  gives:

$$\mu = \underline{P}^t \underline{\lambda} / \underline{P}^t \underline{\delta} \quad (6.6)$$

Equation (6.5) shows that, if a solution to (6.1) exists, then for such a design the deflections  $\underline{\delta}$  are proportional to the dual variables  $\underline{\lambda}$ .

This property of maximum stiffness structures has already been mentioned in the previous section. If a maximum stiffness design has values of design variables denoted by  $\underline{Q}^*$ ,  $\underline{t}^*$  and  $L^*$ , then it can be shown that the inequalities in the dual problem (4.1) are satisfied as equalities for  $\underline{\theta}^j$  corresponding to  $\underline{Q}^*$ . Thus:

$$\underline{\delta}^t \underline{k}^i(\underline{\theta}_j^i) \underline{\lambda} = A_i \quad i=1, \dots, N.$$

Using (6.5) and (6.6) we obtain:

$$\frac{1}{A_i} \frac{\underline{P}^t \underline{\lambda}}{\underline{P}^t \underline{\delta}} \underline{\delta}^t \underline{k}^i(\underline{\theta}_j^i) \underline{\delta} = 1.$$

Now  $\underline{P}^t \underline{\delta} = E$  is the total strain energy of the structure, and  $\underline{P}^t \underline{\lambda} = W(\underline{\delta})$  is the volume of the optimal structure. Thus, we can write:

$$\frac{1}{A_i} (\underline{\delta}^t \underline{k}_i(\underline{\theta}_j^i) \underline{\delta}) = \frac{E_j^i}{W_i} = \frac{E}{W} = \text{Constant}$$

where  $W_i$  is the volume of the  $i$ 'th layer. In other words the strain energy per unit volume within all layers in every element is constant and equal to the strain energy per unit volume of the structure as a whole. This relationship is derived by Venkayya et al in ref. 4 by another approach, and in fact forms the basis of their optimality criterion algorithm. It is a general property of maximum-stiffness structures and is not limited to those which are multilaminar. But it will be shown below that the constant strain energy density condition necessarily cannot always be satisfied, because of the existence of physically infeasible regions in  $\underline{\delta}$ -space. This problem is shown up very clearly by the deflection space formulation.

Relationship (6.5) suggests the following iterative procedure for solving (6.1):

$$\underline{\delta}^{k+1} = \underline{\delta}^k + s^k (\underline{\lambda}^k - \mu^k \underline{\delta}^k) \quad (6.7)$$

$s^k$  is a step-length whose determination will be discussed below. Clearly, when (6.5) is satisfied equation (6.7) will imply no further change in  $\underline{\delta}$ . That iteration (6.7) implies downhill directions on the function  $W(\underline{\delta})$  is easily shown as follows. Downhill directions will be defined as vectors satisfying the following condition:

$$\underline{d}^t \nabla W(\underline{\delta}) \leq 0 \quad (6.8)$$

using equation (3.2) to eliminate the derivative and taking the direction defined by equation (6.7), we then require to prove the following inequality for the  $k$ 'th iteration (the  $k$  is dropped for simplicity)

$$-(\underline{\lambda} - \mu \underline{\delta})^t \underline{K} \underline{\lambda} \leq 0$$

and using (6.6) this would imply:

$$\frac{\underline{P}^t \underline{\lambda}}{\underline{P}^t \underline{\delta}} \underline{\delta}^t \underline{K} \underline{\lambda} - \underline{\lambda}^t \underline{K} \underline{\lambda} \leq 0$$

Using  $\underline{P} = \underline{K} \underline{\delta}$  we obtain

$$\frac{\underline{\delta}^t \underline{K} \underline{\lambda}}{\underline{\delta}^t \underline{K} \underline{\delta}} \underline{\delta}^t \underline{K} \underline{\lambda} - \underline{\lambda}^t \underline{K} \underline{\lambda} \leq 0$$

i.e. we must prove that:

$$(\underline{\delta}^t \underline{K} \underline{\lambda})^2 \leq (\underline{\delta}^t \underline{K} \underline{\delta}) (\underline{\lambda}^t \underline{K} \underline{\lambda}). \quad (6.9)$$

Since  $\underline{K}$  is, by the well known property of stiffness matrices, at least positive semidefinite, it can be written in the form:

$$\underline{K} = \underline{Q}^t \underline{Q}.$$

Define the following variables:

$$\underline{x} = \underline{Q} \underline{\delta}$$

$$\underline{y} = \underline{Q} \underline{\lambda}$$

substituting these expressions in (6.9) we obtain:

$$(\underline{x}^t \underline{y})^2 \leq (\underline{x}^t \underline{x}) (\underline{y}^t \underline{y}).$$

This is the Schwartz inequality, which holds for any vectors  $\underline{x}$ ,  $\underline{y}$  in Euclidian vector space; thus the assertion is proved.

The equality holds in either of two cases:

$$(i) \quad \underline{\lambda} = \mu \underline{\delta}, \quad \text{i.e. at an optimum}$$

$$(ii) \quad \underline{K} \underline{\lambda} = 0.$$

In the case (ii) the stiffness matrix is singular, that is, has at least one zero eigenvalue, which has an eigenvector coinciding with the vector of dual variables,  $\underline{\lambda}$ .

The simplicity of iteration (6.7), as well as its downhill property, recommends its use as the basis of an algorithm for finding maximum-stiffness structures. The following is such an algorithm.

**Step 0** Set  $k = 0$ . For an arbitrary design, generate its deflections  $\underline{\delta}^k$  and its strain energy  $\underline{P}^t \underline{\delta}^k = E$ .

**Step 1** Compute  $W(\underline{\delta}^k)$ ,  $\underline{\lambda}^k$ ,  $\mu^k$ ,  $\underline{d}^k = \underline{\lambda}^k - \mu^k \underline{\delta}^k$ , and  $\nabla W(\underline{\delta}^k) = -\underline{K}^k \underline{\lambda}^k$  where  $\underline{K}^k$  is the stiffness matrix of the optimal design having deflections  $\underline{\delta}^k$ .

**Step 2** If  $|\underline{d}^k| \leq \epsilon_1$  or  $|\underline{d}^{k,t} \nabla W(\underline{\delta}^k)| \leq \epsilon_2$  (where  $\epsilon_1$  and  $\epsilon_2$  are small positive

numbers), then:

Step 2(a) Set  $\underline{\delta}^* = \underline{\delta}^k$ ,  $W^* = W(\underline{\delta}^k)$ ,  $t^* = \underline{t}^k$ ,  $\underline{\theta}^* = \underline{\theta}^k$  and  $L^* = L^k$ : stop. Otherwise:

Step 3 Using for example a parabolic unidirectional search technique as in ref. 5, find  $s^k$  in equation (6.7) such that:

$$W(\underline{\delta}^{k+1}) < W(\underline{\delta}^k)$$

Step 4 If  $|s^k| < \epsilon_3$  (small positive number), go to step 2(a). Otherwise set  $k = k+1$  and return to step 1.

Figures 6.1 and 6.2 illustrate this algorithm. For clarity the plane of diagram 6.2 is chosen to coincide with the hyperplane of constant energy, on which lie all points generated by the above algorithm. For this reason the coordinates axes in the diagram are not the deflections but a linear combination of them.

It is important to realise that this algorithm can converge in a number of ways. Firstly,  $|\underline{\delta}^{k+1} - \underline{\delta}^k|$  can become very small in either of the two ways listed above, that is, either at an optimum or at a point where  $\underline{K}$  is singular and  $\underline{\lambda}$  is an eigenvector of zero eigenvalue. There is, however, another very significant way in which it can stop, namely, when the search directions terminate on the boundary of a physically infeasible region. These regions constitute, in effect, additional constraints on problem P1 over and above the arbitrary constraints  $f_i(\underline{\delta}) \leq 0$ . They are unfortunately not expressible analytically (or do not seem to be) so that they cannot be easily included in the problem statement. However, their existence implies the possibility that the 'solution' (the point where  $\underline{\lambda} = \underline{\mu} \underline{\delta}$ ) does not lie in a physically feasible region: does not, in fact exist. In such a case, the algorithm can be expected to approach the boundary of the physically feasible region in which it has started and finally to halt when the criterion  $|s^k| < \epsilon_3$  is satisfied. This means that a point is generated such that either  $W(\underline{\delta})$  increases sharply after an initial decrease along the search direction, or that a significant move along this direction takes the search into the physically infeasible region in which  $W(\underline{\delta})$  is defined to be infinite. There seems to be some evidence to suggest that the function may in fact tend to increase steeply as the boundary is approached from the physically feasible side, thus forming a natural barrier; but this point remains to be investigated. At any rate, such a 'non-optimal' convergence may indeed be the best that any algorithm could be expected to do, and that it may be a perfectly satisfactory solution will be demonstrated by one of the numerical examples given in section 7 of this paper.

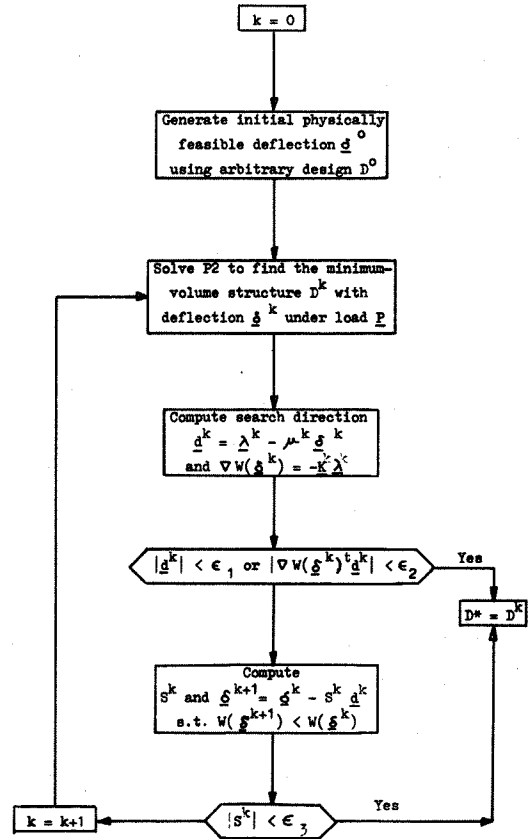


Figure 6.1

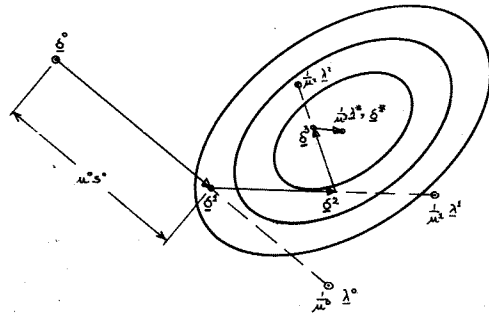


Figure 6.2

## 7. Numerical examples

### An illustrative example

There is a simple truss structure which illustrates very well the techniques described above (fig. 7.1). The problem is to transmit a load  $P = (P_x, P_y)$  from a point A to an infinite foundation unit distance below it by means of pin-jointed bars. The number of bars, their cross-sectional areas, and their angles  $\theta_i$  must be found in such a way as to produce a structure of maximum stiffness.

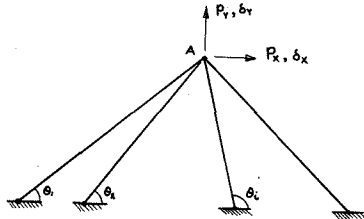


Figure 7.1

These variables are analogous to numbers of layers, thicknesses, and fibre angles in a one-finite-element sheet.

The P1 problem is to find the structure having the lowest possible value of  $W(\delta)$  subject to the condition that  $\underline{P}^t \delta$  is some fixed value (the actual value being arbitrary). It is necessary, for any deflection  $\delta = (\delta_x, \delta_y)$  at A, to be able to solve the P2 problem, which is, to find the structure of minimum volume having that deflection under the given load. We will, for the purposes of illustration, choose the case of horizontal load only ( $P_y = 0$ ;  $P_x = 1000$  E); in addition, rather than allowing an infinite choice of bars, a finite number having attachments to earth at a discrete number of points will be used. This is absolutely not a necessary restriction and indeed ref. 5 shows how the problem can be solved exactly; but it reduces the Functional Linear Programming Problem to one of ordinary Linear Programming. In fact, sixteen bars have been allowed, and the P2 problem is thus: for any  $(\delta_x, \delta_y)$ :

$$\begin{aligned} & \text{Min } \sum_{i=1}^{16} A_i / \sin \theta_i \\ \text{s.t. } & \sum_{i=1}^{16} A_i \left\{ \cos^2 \theta_i \sin \theta_i \cdot \delta_x \right. \\ & \quad \left. + \cos \theta_i \sin^2 \theta_i \cdot \delta_y \right\} = 1000 \quad \text{P2} \\ & \sum_{i=1}^{16} A_i \left\{ \cos \theta_i \sin^2 \theta_i \cdot \delta_x \right. \\ & \quad \left. + \sin^3 \theta_i \cdot \delta_y \right\} = 0. \\ & A_i \geq 0 \\ & 0 \leq i \leq 16 \end{aligned}$$

This is a Linear Program with 16 variables and 2 constraints ( $m = 2$ ); it follows that if it has a solution, such a solution will not have more than 2 nonzero values of  $A_i$ . Hence, since the overall solution is a solution to P2 for some choice of  $(\delta_x, \delta_y)$ , it follows that the overall solution is also a two-bar truss. Notice that this is true no matter how many bars had been allowed as candidates; this is of course the basis of the limits described in section 5.

Problem P2 was solved for a range of values of  $(\delta_x, \delta_y)$ ; the results are shown in figure 7.2. This should be regarded as the upper half of a symmetrical graph. There is no solution to P2 for deflections on the hatched side of line A-A; that is, there are no structures having such

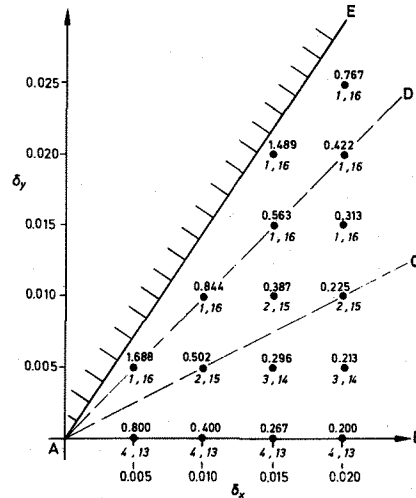


Figure 7.2

deflections under horizontal loads. The P1 problem can now be expressed as follows: find the point on the  $(\delta_x, \delta_y)$  plane such that  $W(\delta)$  is a minimum subject to  $P_x \delta_x = \text{constant}$ . That is, we must find the minimum of  $W(\delta)$  along any arbitrary vertical line on the diagram. Inspection quickly reveals that such a point will lie on the  $\delta_x$  axis, although of course such a conclusion would have to be verified by solving P2 for a wider range of values of  $(\delta_x, \delta_y)$ . All designs along the  $\delta_x$  axis can be seen to have the same layout; indeed this is true for any line passing through the origin and lying in the physically feasible region. This layout is orthogonal, and is in fact the Michell solution to the problem.

#### Two Composite Sheets

The algorithm developed in section 6 was implemented as a fortran program for the DEC PDP-10 computer at Hatfield Polytechnic, and a number of problems have been solved using it. Two of these are shown below (figures 7.3, 7.4). They are a cantilever structure and a sheet with a circular cutout, subject to simple tensile stress.

#### Cantilever

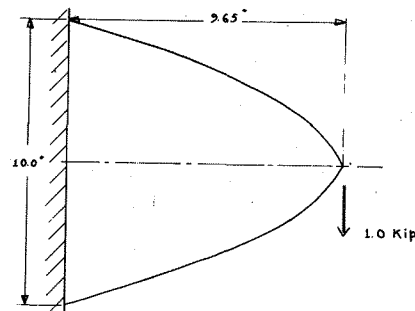


Figure 7.3

The overall form of the cantilever was chosen



to approximate a Michell structure, and the finite element mesh was defined so as to follow approximately the lines of principal stress in such a structure. The results are fully reported in refs 5 and 7; only the highlights will be discussed here.

The problem was solved from two distinct starting points. The first was a design of unit thickness, with a single layer in each finite element with fibres parallel to the centre line. The second had fibres at an angle of  $-0.8$  rads above the centre line and  $+0.8$  below it, the angles being measured anticlockwise from that line.

The overall results of the runs from these two starting points are summarised in table 1; all

	1	2	3	4	5
S.P.1	37.6940	3.5395	3.1357	0.002	5
S.P.2	11.7062	4.4000	3.1355	0.001	6

Table 1

1. Initial volume (cubic inches)
2. Initial optimum volume (fixed deflection)
3. Final optimum volume
4.  $|\underline{\delta} - \lambda/\mu|/|\underline{\delta}|$
5. Iterations.

volumes in the second row of this table have been scaled so that both starting points had the same strain energy. This ensures that solutions are comparable.

The first column in the table is self-explanatory. The second column is the volume of the structure having the same deflection as the starting design, but of minimum volume. The ratios of these volumes is remarkable; in the first case, more than 10 : 1. Column 2 was obtained by solving one Functional Linear Programming problem, i.e. P2, in each case. The final volumes, shown in column 3, were obtained by the algorithm of section 6. As can be seen, the same volume was obtained from both starting points; in fact, the designs were virtually identical. Column 4 gives a measure of the optimality of the designs; it shows the angle between the virtual deflection  $\lambda$  and the actual deflection  $\underline{\delta}$ , which would be zero for a true maximum stiffness design. It can be seen that this is very nearly the case.

The actual design achieved is discussed in detail in refs. 5 and 7. It turned out that the fibres more or less followed the principal stress lines of a Michell structure, and that considerable numbers of the finite elements ended up with orthogonal fibre layouts. The arguments of section 6 show that no element can have more than three layers in this case; most in fact have either one or two.

#### Sheet with Cutout (Figure 7.4)

This problem was in one sense a more difficult one than the cantilever, because it does not appear to possess a true solution, in the sense that no structure seems to exist having a uniform

strain-energy density. Since the algorithm being used actively seeks such a design, it follows that it will finish in a non-standard way.

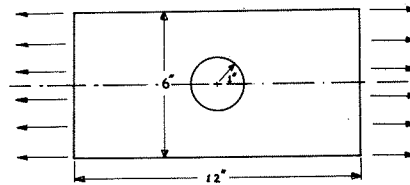


Figure 7.4

1	2	3	4	5
17.227	14.267	12.683	0.57	11

Table 2

1. Initial volume (cubic inches)
2. Initial optimum volume (fixed deflection)
3. Final optimum volume
4.  $|\underline{\delta} - \lambda/\mu|/|\underline{\delta}|$
5. Iterations.

Table 2 shows the main results of the run; once again the final design is discussed in detail in Refs. 5 and 7. It can be seen that column 4 of table 2 indicates non-optimality in the strict sense. In fact, the algorithm finally stopped on the boundary of the physically feasible region, after 11 iterations; there was no evidence of erratic behaviour or other difficulty (see ref. 5 for a more detailed discussion). Clearly, the necessity of stopping near the physically infeasible region is not a fundamental source of difficulty, which is encouraging since the existence of such regions introduces difficulties in analysis.

The problem has been solved by Khot et al<sup>(4)</sup> who found a design about 18% heavier. Since the mesh used by them was much finer than the present one, comparisons are difficult; however, it does suggest that the solution obtained by the deflection-variable technique was a valid one.

#### 8. Conclusions

This paper has described an approach to the optimisation of multilaminar composites. The innovation involved consists in viewing the problem in terms of deflections rather than design variables, and it reveals an underlying structure which is not obvious in other formulations. By exploiting this structure, it has been shown that algorithms can be designed which allow numbers of layers, their thicknesses and their fibre orientations to be optimised simultaneously; in addition, useful general insights can be gained into the form of optimal composites. Indeed, this approach to structural optimisation can obviously be extended to other structural systems, both as an analytical and a computational tool.

Although the methods described in this paper are clearly still at an early stage of development, it is worth considering the contribution which they might, with very little extra effort, be able to

make to the practical design of fibre-reinforced laminates. Their first advantage is the freedom which they allow in the choice of fibre angles. One can imagine the functional linear programming algorithm initially being used in conjunction with one of the optimality criterion methods in the following way. As has already been mentioned, such methods, of necessity, restrict the allowable fibre angles to a small set. However, if  $\delta^*$  is the deflection of a design found by an optimality-criterion method, the FLP algorithm could be used to find the minimum-volume design corresponding to this deflection, without restriction on the fibre angles, or with fewer restrictions. The result may not be truly optimal, since  $\delta^*$  will not be an optimal deflection, but it is likely that it will be an improvement over the first attempt. In this way the FLP algorithm could be further tested and developed; at some stage algorithms for the solution of the outer subproblem could be introduced and tested in parallel with the established methods.

The solution to the problems described in section 7 had one characteristic which would often be undesirable in practice; namely, finite elements which were "empty" in the sense of having no layers assigned to them. Normally some lower limit on the total thickness of material in any element would be a part of the problem statement. Such a constraint can be very easily applied as part of the functional linear programming algorithm; empty elements are not an inherent feature of the method. However, the fact that the algorithm can, if allowed, point to areas of the structure which perhaps ought to be empty is a useful feature because it suggests an interactive way of using the method for shape optimisation. Suppose, for example, that the cantilever structure of problem 2 had been defined, not by a curve, but by a rectangular envelope containing it. An initial application of the deflection-variable algorithms would be expected to produce empty elements around the upper and lower free corners. The FLP algorithm automatically prevents any node from becoming isolated, however, so that some vestigial elements would remain in these regions to allow all nodes to remain connected. A next step would be to redefine the finite element mesh to eliminate nodes which clearly served no purpose, thus allowing larger areas to be emptied on the next run. In this way the optimum outline, as well as the internal design, of the structure would emerge. This had been done in part for the finite element mesh of figure 7.5, where early experiments had shown that the root area between the two (final) points of support ought to be empty and was therefore not assigned an element.

According to some authors, e.g. Ham and Willshire<sup>(10)</sup> the form in which fibre-reinforced composites will be initially used on a large scale in aeronautical structures will be as reinforcing on isotropic primary structures. Mixed problems of this type would present no difficulty for the methods described in this report; the P2 problem would simply take on a form with some constraint columns,  $p_1$ , fixed and some variable. The functional linear programming algorithm is capable of solving such problems without modification.

Finally, any discussion of the scope of the deflection-variable method would not be complete without a reference to its generality; it is not

just applicable to multilaminar structures. Problems involving trusses, space frames, rigid frames and isotropic sheets can also be approached in this way. The inner subproblem would differ in its form from one class of structures to another; in some cases it would be of classical L.P. form, and in others FLP. This would seem to be an interesting area for further research.

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